Notes on Neron Models

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Abstract

In these notes, we discuss the theory of Neron models and compare it with the theory of relative Picard functors. These notes are based on a talk that the author gave at a student seminar at Harvard in winter 2007.

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1 The Neron Model

The theory of Neron models is motivated by the problem of degenerating abelian varieties. Suppose that S is the spectrum of a discrete valuation ring. Given an abelian variety A_t over the generic point of S, a natural question to ask is "how to extend the abelian variety A_t to a family over all of S?"

In general, it is too much to ask that the family be extended in such a way that the central fiber is an abelian variety. The idea of Neron is to try to extend the family over the central point in such a way that the central fiber is a smooth group variety, but to allow the central fiber to be non-proper.

Neron's theory has had a number of applications in algebraic geometry and number theory. These applications include:

- The theory of stable reduction of abelian varieties and stable reduction of curves
- The Birch-Swinnerton-Dyer conjecture
- The theory of heights in diophantine geometry
- Lucia Caporaso's work on the Neron model of the universal Jacobian
- Recent work of Jason Starr and Tom Graber

In these notes, we review some of the basic theory of Neron models and discuss their relationship with the relative Picard functor. Much of our treatment of these topics is based on the book [3] and the article [10]. In particular, most of our examples are taken directly from Raynaud's article.

Terminology and Notation. For the remainder of this note, we will work over a fixed discrete valuation ring that we denote by S. Two important examples to keep in mind are the formal analytic disc, $\mathrm{Spec}(\mathbb{C}[[z]])$, and the spectrum of the p-adic numbers $\mathrm{Spec}(\mathbb{Z}_p)$. The special point of S will be denoted by s, while the generic point will be denoted t. The residue fields at these points are denoted by $k(s)$ and by $k(t)$ respectively. Set $k(\bar{t})$ equal to a fixed algebraic closure of $k(t)$ and let $\bar{t}$ denote the spectrum of this field. We will frequently write $k(s)$ for $\mathrm{Spec}(k(s))$ and similarly for $k(t)$. In these notes, an **abelian variety** over a field k is a proper, smooth, and geometrically connected k-group scheme that is of finite type. We also need to define what is meant by a family of curves. If T is a base scheme, then a **relative T-curve** is a flat, proper morphism $X \to T$ with geometric fibers that are equidimensional of dimension 1. There is some disagreement over the appropriate definition of a projective morphism. We will say that a morphism $f : X \to T$ is **projective** if X is isomorphic to a closed subscheme of $\mathbb{P}(\mathscr{E})$ for some quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$ that is of finite type. This definition coincides with the usage of the term in [5]. The relevant sections are 5.3 and 5.5.

Remark. In these notes, we will be working exclusively work over a base S that is the spectrum of a discrete valuation ring. Much of the theory remains true if the base S is allowed to be an arbitrary Dedekind scheme. It is, however, necessary that the base be 1-dimensional and regular. There has been some work on extending the theory of Neron models to more complicated base schemes, but serious issues arise. Some discussions of issues related to extending the theory of Neron models can be found in Deligne's article [4].

With the notation out the way, let us define Neron models:

Definitions. Suppose that $A_t/k(t)$ is an abelian variety.

A **model** of the abelian variety $A_t/k(t)$ is an S-scheme A/S together with an identification of the generic fiber of A with A_t.

A **Neron model** of an abelian variety $A_t/k(t)$ is an S-model A/S of $A_t/k(t)$ that is smooth and satisfies the Neron mapping property: for every smooth morphism $T \to S$ and a $k(t)$-morphism $u_t : T|_t \to A_t$, there is a unique S-morphism $u : T \to A$ that extends $u|_t$.

Remarks.

- The definition of a Neron model does not make use of the fact that $A_t/k(t)$ is an abelian variety. One can define the Neron model of an arbitrary smooth $k(t)$-scheme, but this definition does not seem to have nice properties in general. For example, one can show that $\mathbb{P}^1_S/S$ is not the Neron model of its generic fiber.

- The definition of the Neron model does not make explicit reference to the group scheme structure of the Neron model. It follows from general formalism that a Neron model has canonical S-group scheme structure. For example, the multiplication law is constructed by applying the Neron mapping property to $T = A \times_S A$ and u_t equal to the group law $m : A_t \times_{k(t)} A_t \to A_t$ of the abelian variety. It also follows from standard formalism that the Neron model of $A_t/k(t)$ is unique up to a unique isomorphism.

- The Neron mapping property should be thought of as a weakening of the valuative criteria of properness. For any S-scheme A/S, there is a natural map $A(S) \to A(k(t)) = A_t(k(t))$. If A/S is either proper or satisfies the Neron mapping property, then this map is bijective. In other words, both of these conditions imply that every rational section of $A \to S$ extends uniquely to a regular section. However, the Neron mapping property does NOT imply that an arbitrary rational multi-section can be extended to a regular multi-section. By contrast, this stronger extension property is a consequence of the valuative criteria of properness.

- Unlike many algebro-geometric objects that are defined by universal properties, the Neron model does not commute with arbitrary extensions of the base S. In other words, if $S' \to S$ is an arbitrary extension of discrete valuation rings, then it need not be the case that $A_{S'}$ is the Neron model of its generic fiber. One can show that $A_{S'}$ is the Neron model of its generic fiber if $S' \to S$ is faithfully flat and the ramification index of S' over S is 1 (so a uniformizer of S maps to a uniformizer of S'). This assumption is weaker than the assumption that $S' \to S$ is étale since we are allowing the residue extension of S'/S to be inseparable.

The basic existence theorem is:

Theorem 1. *The abelian variety $A_t/k(t)$ admits a Neron model that is S-separated and of finite type over S.*

Proof. There are two main tools that are used in the proof. The first tool is a smoothening process. Given an arbitrary S-model for $A_t/k(t)$, Neron gives a procedure for modifying the given model into a smooth S-model. The second tool is a general result on extending birational group laws to genuine group laws. We refer the interested reader to the literature for details.

Neron's original paper is [9]. A detailed modern treatment of the theorem can be found in [3]. Another good article on Neron models is Artin's article [2]. □

Remark. In his original paper [9], Neron stated the above theorem with the additional hypothesis that the residue field $k(s)$ is perfect. In [3], it is shown that this hypothesis can be removed.

2 The Relative Picard Functor

Jacobians of smooth curves provide an important source of abelian varieties. Given a curve $X_t/k(t)$ over the generic point of S, the Neron model provides one method for degenerating the Jacobian of X_t. An alternative approach would be to first extend the curve $X_t/k(t)$ to a family of curves over S and then try to degenerate the Jacobian in such a way that reflects the geometry of the family of curves. This idea leads us to the relative Picard functor.

Fix a S-curve $f : X \to S$. The key definition is the following:

Definition. The **relative Picard functor** $\mathrm{Pic}_{X/S} : (\text{S-sch})^{\mathrm{op}} \to \text{sets}$ is defined to be the fppf sheaf associated to the presheaf given by the rule $T \mapsto \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$. Here Pic denotes the set of isomorphism classes of line bundles. The set $\mathrm{Pic}(T)$ is considered a subset of $\mathrm{Pic}(X_T)$ via pullback.

There are a number of delicate technical issues that arise with this definition. For example, it is not necessarily true that every element of $\mathrm{Pic}_{X/S}(T)$ can be represented by a line bundle on X_T. In these notes, we omit any detailed discussion of these issues and direct the interested reader to Kleiman's excellent article [8].

An important subsheaf of $\mathrm{Pic}_{X/S}$ is the subsheaf $\mathrm{Pic}^0_{X/S}$ that parameterizes degree 0 line bundles. When S is the spectrum of an algebraically closed field and X/S is a smooth curve, the sheaf $\mathrm{Pic}^0_{X/S}$ is represented by the Jacobian of the curve.

We will be primarily interested in the geometry of $\mathrm{Pic}^0_{X/S}$ rather than the geometry of $\mathrm{Pic}^0_{X/S}$. We will often be sloppy and refer to $\mathrm{Pic}^0_{X/S}$ as the relative Picard scheme, although strictly speaking it is a component of the relative Picard scheme.

There are two basic existence theorems for the relative Picard functor:

Theorem 2 (Grothendieck). *If $f : X \to S$ is flat, projective, and has 1-dimensional integral geometric fibers, then $\mathrm{Pic}_{X/S}$ can be represented by a smooth, separated S-group scheme that is locally of finite type. Furthermore, the subscheme $\mathrm{Pic}^0_{X/S}$ is of finite type.*

Proof. The original proof can be found in [7]. A modern exposition of the proof can be found in [8]. □

Theorem 3 (Artin). *If $f : X \to S$ is proper, flat, cohomologically flat (i.e. $f_*(\mathscr{O}_X) \cong \mathscr{O}_S$ holds universally) and has geometric fibers that are equidimensional of dimensional 1, then $\mathrm{Pic}_{X/S}$ can be represented by a formally smooth S-group space that is locally of finite type.*

Proof. This is a standard application of Artin's representability theorem. The proof can be found in [1]. □

Remarks:

- The hypotheses of Grothendieck's theorem imply that f is cohomologically flat, so the hypotheses of Grothendieck's theorem are strictly stronger than those of Artin's theorem.

- Similarly, the conclusion of Artin's theorem is weaker than Grothendieck's theorem. There are examples of families of curves X/S for which the relative Picard scheme exists only as a non-separated algebraic space and the subspace $\mathrm{Pic}^0_{X/S}$ is not locally of finite type.

- The fact that $\mathrm{Pic}_{X/S}$ is always smooth is particular to the case of curves and essentially follows from the fact that the cohomological dimension of $f : X \to S$ is 1. There are examples of surfaces over fields of positive characteristic for which the Picard scheme is singular.

- The intuition behind the condition of cohomological flatness is that a curve is not allowed to acquire extra global functions under specialization. It follows from Zariski's theorem on formal functions that the special fiber of $f : X \to S$ has the same number of connected components as the generic fiber, so these "extra" global functions can only exists when the special fiber is non-reduced. We will see some examples later.

- Artin's theorem is almost sharp. In section 8.2.1 of [10], Raynaud proves that for a family of curves $f : X \to S$ the representability of $\mathrm{Pic}_{X/S}$ together with some additional mild condition is equivalent to the cohomological flatness of $f : X \to S$. Examples of families of curves $f : X \to S$ such that f is not cohomologically flat, but $\mathrm{Pic}_{X/S}$ is representable are related to certain technical issues related to our definition of the relative Picard functor.

- A detailed discussion of cohomological flatness can be found in section 7 of [10]. In particular, a general criteria for proving cohomological flatness can be found in that article. One consequence of the criteria is the following. Suppose that S is strict henselian and of characteristic $(0, 0)$. Let $f : X \to S$ be a family of curves such the special fiber X_s has no embedded points and that for every minimal prime x of the special fiber X_s the local ring $\mathscr{O}_{X,x}$ of the total space is normal. If $f_*(\mathscr{O}_X) = \mathscr{O}_S$, then $f : X \to S$ is cohomologically flat.

Let's consider some examples. In studying our examples, we want to illustrate geometric content so we work over the complex numbers $\mathbb{C}$. The base S will be the strict henselization of a complex curve at a point. Geometrically, this scheme roughly looks like an arbitrarily small neighborhood of $\mathbb{P}^1_{\mathbb{C}}$ in the classical topology. n our examples, the schemes $Pic^d_{X/S}$ are all isomorphic.

In the examples, we will describe $\mathrm{Pic}^0_{X/S}$ by describing the generic fiber and the special fiber. For our choice of S, many of the technical issues concerning the relative Picard functor simplify greatly. In particular, the special fiber of $\mathrm{Pic}^0_{X/S}$ is always complex group space and the group of $\mathbb{C}$-valued points of this space can be identified with the group of isomorphism classes of degree 0 line bundles on X_s. Similarly, the group of $k(\bar{t})$-valued points of the generic fiber of $\mathrm{Pic}^0_{X/S}$ can be identified with the group of isomorphism classes of line bundles on $X_t \otimes k(\bar{t})$.

We will analyze the relationship between the special fiber of $\mathrm{Pic}^0_{X/S}$ and the generic fiber of $\mathrm{Pic}^0_{X/S}$ by looking at the different ways in which the identity section of the generic fiber can be specialized to the special fiber. We say that a point $g \in \mathrm{Pic}^0_{X/S}(k(s))$ of the special fiber is a **specialization of the identity section of the generic fiber** if there is a section $\sigma \in \mathrm{Pic}^0_{X/S}(S)$ such that $\sigma(t)$ is trivial and $\sigma(s)$ equals g. When $\mathrm{Pic}^0_{X/S}$ is separated over S, the identity of the generic fiber can only be specialized to the identity so the concept is only interesting when $\mathrm{Pic}^0_{X/S}$ is non-separated.

We present four examples in increasing order of complexity.

Example 1. Suppose that $g : Y \to \mathbb{P}^1$ is a general pencil of plane cubics. Such a pencil has a finite number of singular elements and every singular element is a nodal cubic. Let S denote the strict henselization of $\mathbb{P}^1$ at a point corresponding to a nodal cubic and let $f : X \to S$ be the restriction of the pencil to S.

The family $f : X \to S$ satisfies the hypotheses of Grothendieck's theorem, so $\mathrm{Pic}^0_{X/S}$ can be represented by a smooth, separated S-group scheme that is of finite type.

The fibers of $\mathrm{Pic}^0_{X/S} \to S$ can be computed in a number of ways. For example, the generic fiber is the Jacobian of a smooth genus 1 curve and hence an elliptic curve. The special fiber can be computed by taking the normalization of the nodal cubic curve and comparing line bundles on the nodal curve to line bundles on the normalization. Such an argument shows that the special fiber is the multiplicative group $\mathbb{G}_m$.

Example 2: Now suppose that $g : Y \to \mathbb{P}^1$ is a pencil that contains the plane cubic curve given by the union of a line L and a quadric Q, but is otherwise general. Again, let $f : X \to S$ denote the restriction of this pencil to the strict henselization S of $\mathbb{P}^1$ at the point corresponding to the reducible cubic. Because we constructed $f : X \to S$ from a family of plane curves, it follows that $f : X \to S$ is cohomologically flat.

By Artin's theorem, the relative Picard functor $\mathrm{Pic}^0_{X/S}$ can be represented by a formally smooth S-group space that is locally of finite type. It does not follow from the theorems that we have stated, but one can show that the relative Picard functor is in fact represented by a scheme. Let's determine the fibers of $\mathrm{Pic}^0_{X/S}$ and figure out how the generic fiber specializes to the special fiber.

As in the first example, the generic fiber is an elliptic curve. What about the special fiber? Unlike the previous example, the special fiber of $\mathrm{Pic}^0_{X/S}$ has a non-trivial discrete component. Given a line bundle $\mathscr{L}$ of degree 0 on X_s, we can consider the bidegree $(\deg(\mathscr{L}|_L), \deg(\mathscr{L}|_Q))$. The sum of the bidegrees is equal to the total degree, so $\deg(\mathscr{L}|_L) + \deg(\mathscr{L}|_Q) = 0$ but the individual terms can otherwise be arbitrary. The subgroup of $\mathrm{Pic}^0_{X/S}|_s$ consisting of bidegree $(0, 0)$

line bundles can be computed by, for example, comparing these line bundles with line bundles on the normalization. Such a study shows that this group is isomorphic to the multiplicative group $\mathbb{G}_m$. Putting these facts together, we can conclude that the special fiber of $\mathrm{Pic}^0_{X/S}$ is isomorphism to $\mathbb{Z} \times_{k(s)} \mathbb{G}_m$.

Now let's analyze how the generic fiber specializes to the special fiber. The identity of the generic fiber can certainly specialize to the identity of the special fiber. There are, however, other ways of specializing the identity of the generic fiber. Consider the ideal sheaf I_L of the line L in the total space X of the family. The scheme X is regular, so the sheaf I_L is a line bundle. The restriction of this line bundle to the generic fiber X_t is trivial. On the other hand, the restriction of I_L to Q is the ideal sheaf of the two points where Q meets L and hence has degree equal to -2. From the relation $\deg(\mathscr{L}|_L)+\deg(\mathscr{L}|_Q) = 0$, it follows that the restriction of I_L to L is degree 2. In summary, I_L is a line bundle such that the restriction to X_t is trivial and the restriction to X_s has bidegree $(2, -2)$. In particular, this line bundle defines a specialization of the identity of $\mathrm{Pic}_{X/S}|_t$ to a non-trivial line bundle. This shows that $\mathrm{Pic}^0_{X/S}$ is NOT separated. Further consideration shows that for every integer n there is a unique specialization of the identity to a line bundle of bidegree $(2n, -2n)$.

To conclude, the family of curves $f : X \to S$ is a family of curves that satisfies the hypotheses of Artin's theorem. The relative Picard scheme $\mathrm{Pic}^0_{X/S}$ is representable by a scheme that is not separated and not locally of finite type.

Example 3. Take the twisted cubic in $\mathbb{P}^3$ and try to smash the curve into a planar curve. More formally, define a family of space curves $Y \to \mathbb{A}^1$ by setting the fiber over $t \neq 0$ equal to the image of the standard twisted cubic under the automorphism $[W, X, Y, Z] \mapsto [W, X, Y, tZ]$ and then defining the fiber over 0 to be the limit in the Hilbert scheme. One can check that this family can be taken to be given by the equations:

$$t^2(X + W) - Z^2 = tX(X + W) - YZ = XZ - tWY = Y^2 - X^2(X + W) = 0$$

Let S denote the strict henselization of $\mathbb{A}^1$ at the origin and $f : X \to S$ the restriction of $Y \to \mathbb{A}^1$ to S. The generic fiber of this family is the twisted space cubic and hence isomorphic to $\mathbb{P}^1$. The special fiber is more interesting. One can check that the special fiber is a non-reduced curve whose underlying reduced subscheme is a nodal plane cubic. This curve has a unique embedded point at the node. The space of global regular functions $H^0(X_s, \mathscr{O}_{X_s})$ is 2-dimensional and a basis is given by $1, z = Z/W$. In particular, the structure morphism $f : X \to S$ is not cohomologically flat.

The functor $\mathrm{Pic}^0_{X/S}$ is formally smooth, locally finitely presented, and has representable fibers, but we no longer know that $\mathrm{Pic}^0_{X/S}$ is representable. Indeed, it turns out that this functor is not representable. We can still compute the structure of the fibers and analyze how the generic fiber specializes to the special fiber.

The generic fiber of $f : X \to S$ is a smooth curve of genus 0, so the generic fiber of $\mathrm{Pic}^0_{X/S}$ is the trivial group. One can show that the special fiber of $\mathrm{Pic}^0_{X/S}$

is canonically isomorphic to the connected component of the Picard variety of the underlying reduced curve and hence is isomorphic to $\mathbb{G}_m$. This is most easily proven cohomologically. In particular, every line bundle on the special fiber can be written as $\mathscr{O}_{X_s}(p-q)$ for some smooth points p and q of X_s.

Let us now turn to the question as to which points in the special fiber are specializations of the identity of the generic fiber. A typical line bundle $\mathscr{L}$ on X_s can be written as $\mathscr{O}_{X_s}(p-q)$ for smooth points p and q of X_s. Because we are working over a strict henselian base, these points can be extended to sections $\sigma, \tau : S \to X$ respectively. Consider the line $\mathscr{O}_X(\sigma - \tau)$ on the total space. This line bundle restricts to the line bundle $\mathscr{L}$ on the special fiber X_s. The restriction of $\mathscr{O}_X(\sigma-\tau)$ to the generic fiber must be trivial. This establishes that $\mathscr{O}_X(\sigma - \tau)$ induces a section of $f : X \to S$ that is a specialization of the trivial line bundle to $\mathscr{L}$. Since $\mathscr{L}$ was arbitrary, we have established that EVERY line bundle on the (1-dimensional) special fiber of $\mathrm{Pic}_{X/S}$ is the specialization of the identity. The relative Picard functor $\mathrm{Pic}^0_{X/S}$ is not locally separated and hence not representable.

Example 4. Our fourth example is a family of non-reduced curves. Let $E/\mathbb{C}$ be a fixed elliptic curve with origin O. We can consider the product $E \times_{\mathbb{C}} E$ as a constant family of elliptic curves over the curve E via the second projection map. Let S be the strict henselization of the curve E at the origin and let $f : X_0 \to S$ be the restriction of the constant family of E to S. The example that we will examine is constructed from the family $X_0 \to S$. To construct this family, we need an auxiliary line bundle. Define $\mathscr{L}$ to be the line bundle $\mathscr{O}(\Delta - O \times_{\mathbb{C}} E)$. Here Δ is the diagonal of $E \times_{\mathbb{C}} E$. The scheme X is defined to be the scheme whose underlying topological space is equal to the underlying topological space of X and whose structure sheaf is equal to $\mathscr{O}_X = \mathscr{O}_{X_0} \oplus \mathscr{L}$. Here the ring structure on the direct sum is defined so that $\mathscr{L}$ is a square-zero ideal in $\mathscr{O}_X$. It is immediately verified that $X \to S$ is proper, flat, and has 1-dimensional geometric fibers.

The family $X \to S$ can be visualized as follows. An element of this family is a 1-st order neighborhood around the elliptic curve E. For a generic element of this family, the 1-st order neighborhood is twisted in such a way that there are no non-zero global functions that vanish at every point. In the special fiber, the 1-st order neighborhood becomes untwisted in such a way that there is a 1-dimensional space of global functions that vanish at every point but have non-zero derivatives.

The fibers of the relative Picard functor can be computed in terms of cohomology of $\mathscr{L}$. Consider the natural map $\mathrm{Pic}^0_{X_s/k(s)} \to \mathrm{Pic}^0_{E/k(s)}$ from the Picard scheme of X_s to the Picard scheme of the underlying reduced subscheme of X_s. A standard computation shows that this map is surjective and the kernel is equal to the vector space scheme associated to $H^0(X_s, \mathscr{L})$. The analogous result holds on the geometric generic fiber as well. The fiber-wise cohomology of $\mathscr{L}$ is $h^0(X_t, \mathscr{O}_{X_t}) = 0$ and $h^0(X_s, \mathscr{O}_{X_s}) = 1$. In particular, the group variety $\mathrm{Pic}^0_{X_{\bar{t}}/k(\bar{t})}$ is isomorphic to the an elliptic curve while the group variety

$\mathrm{Pic}^0_{X_s/k(s)}$ is an extension of an elliptic curve by the additive group $\mathbb{G}_a$. One can show that scheme-theoretic closure of the identity in the special fiber of $\mathrm{Pic}^0_{X/S}$ is the subgroup $\mathbb{G}_a$. As in example 3, it follows that the relative Picard functor is not locally separated and hence not representable.

3 The Comparison Theorem

How does the relative Picard functor compare to the Neron model in the four examples from the previous section? In the first example the relative Picard scheme looks like a reasonable candidate for the Neron model of the generic fiber, while in the second, third, and fourth examples the relative Picard functor can not be isomorphic to the Neron model. The second example can not be a Neron model because the relative Picard scheme is not S-separated, while the third and fourth examples were not even representable by an algebraic space.

In the last three examples, an obvious obstruction to the relative Picard scheme being the Neron model, or more generally having nice properties, is the lack of separatedness. It is natural to consider instead the **maximal separated quotient** of $\mathrm{Pic}^0_{X/S}$.

Definitions. Suppose that $f : X \to S$ is proper, flat, and has equidimensional fibers of dimension 1. We define the S-group functor E/S to be the **scheme-theoretic closure of the identity** in $\mathrm{Pic}^0_{X/S}$. This is defined to be the fppf subsheaf of $\mathrm{Pic}^0_{X/S}$ generated by those points of the form $z \in \mathrm{Pic}^0_{X/S}(Z)$ with the property that $z_t \in \mathrm{Pic}^0_{X/S}(Z_t)$ is trivial and $Z \to S$ is flat. One can check that if $\mathrm{Pic}^0_{X/S}$ is representable, then this coincides with the usual notion of scheme-theoretic closure.

The **maximal separated quotient** of $\mathrm{Pic}^0_{X/S}$ is defined to be the fppf quotient sheaf $\mathrm{Pic}^0_{X/S}/E$ and is denoted $Q^0_{X/S}$.

Remark. The process of taking scheme-theoretic closure is particularly simple for families of schemes over a discrete valuation ring and the given functorial definition of scheme-theoretic closure is particular to this case. Suppose that $P \to S$ is a morphism from an arbitrary scheme to the spectrum of a discrete valuation ring and that E_t is a closed subscheme of the generic fiber. The scheme-theoretic closure $\bar{E}_t$ of E_t in P is characterized by the fact that it is S-flat and the generic fiber is equal to E_t. A discussion of these issues can be found in [6] (EGA, IV, (2.8.5)).

The maximal separated quotient $Q^0_{X/S}$ has somewhat nicer properties than the relative Picard functor:

Theorem 4. *Suppose that $f : X \to S$ is proper, flat, and has equidimensional geometric fibers of dimension 1. Then maximal separated quotient $Q^0_{X/S}$ is a smooth, separated S-group scheme that is of finite type.*

Proof. This is proposition 8.0.1 in Raynaud's paper [10]. □

We have the following result comparing the Neron model to the relative Picard scheme:

Theorem 5. *Suppose that $f : X \to S$ is proper, flat, and has equidimensional fibers of dimension 1 and that the total space X is regular. Suppose further that either $k(s)$ is perfect or $f : X \to S$ admits a section. Then $Q^0_{X/S}$ is the Neron model of its generic fiber.*

Proof. This is theorem 8.1.4 in Raynaud's paper [10]. □

Remarks: The hypotheses of the theorem can be weakened, but then become somewhat more difficult to state. We refer the interested reader to Raynaud's paper.

Let's revisit the examples from the previous section in light of the previous two theorems.

Example 1. It follows from theorem theorem 5 that $\mathrm{Pic}^0_{X/S}$ is the Neron model of its generic fiber.

Example 2. It follows from the theorem that $Q^0_{X/S}$ is the Neron model of its generic fiber. From our earlier discussion of this example, we can see that the special fiber of $Q^0_{X/S}$ is isomorphic to $\mathbb{G}_m \times \mathbb{Z}/2\mathbb{Z}$.

Example 3. In this example, the total space X is not regular so we can no longer use theorem 5 of compare $Q^0_{X/S}$ to the Neron model of the generic fiber. We can still compute $Q^0_{X/S}$. It follows from our earlier discussion that $Q^0_{X/S}$ is just the trivial S-group scheme.

Example 4. As in example 3, the total space X is not regular so we can not cite theorem 5, but we can compute $Q^0_{X/S}$. In this case, the group scheme $Q^0_{X/S}$ is a family of elliptic curves.

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